Final Exam
(25 points)

On a two-century world, unconstrained behavior by each century's aggregated population has each applying symmetric preferences according to

$$U_1 = x_1^{\alpha} y_1^{\beta} \quad \text{and} \quad U_2 = x_2^{\beta} y_2^{\alpha} \quad \text{where } \alpha >> \beta > 0 \text{ } (\alpha \text{ is much larger than } \beta).$$

1 denotes Centurion 1 (or C1): the community of all people living in the 1st century.

2 denotes Centurion 2 (or C2): the community of all people living in the 2nd century.

x_1 is consumption by C1 out of its own production (q_1).

y_1 is gifted by C1 to C2, out of C1's production (q_1).

x_2 is gifted by C2 to C1, out of C2's production (q_2).

y_2 is consumption by C2 out of its own production (q_2).

[The possibility that each Centurion has some regard for the other ($\beta>0$) is a reasonable consequence of two factors: mild altruism and some overlap in that some people are members of both C1 and C2. Notice that no one's utility is changed as a result of gifts received; assume this is a fact.]

An identical, decreasing marginal product production function applies for each period,

$$q_i = m_i^k; \; i=1,2; \; k<1; \; \text{with } m_1 + m_2 \leq \bar{M}$$

where m is the only input. k and $\bar{M}$ are constants. The m_i are endowments – basicly the legal or ethical division of the sole input across the two centuries.

Except for gifting, trade between C1 and C2 is impossible, so $x_i+y_i=q_i$ and these 3 commodities necessarily have the same intra-century value (p_i). In general, $p_1 \neq p_2$ is expected.

(10 pts) a. For C1 only and for arbitrary α, β, k, and m_1, find the competitive equilibrium resulting from optimizing behavior by 2 agents. One C1 agent is the producer of q_1, and the other C1 agent is the aggregate buyer of x_1 and y_1 with the preferences stated above. The buyer/consumer is the only property owner. Neither agent exerts noncompetitive market power.

(2 pts) b. Accurately state the similar competitive equilibrium resulting for C2.

(5 pts) c. Are the results of (a) and (b) efficient regardless of the m_i, including the case where $m_2=0$ (which would occur if C1 claims all $\bar{M}$ because C2 is not present yet)? Discuss.

(2 pts) d. Under what conditions would the equilibria found in parts (a) and (b) result in $p_1=p_2$?

(6 pts) e. If a social planner applied standard discounting tools to study optimal inter-century allocation, a social welfare function like the following might be applied. In light of your prior findings, what control variable could the planner employ to achieve maximum W, and what would the result be like? Please comment on the normative issues here.

$$W = U_1 + U_2^{1/(1+d)^{100}} \quad \text{with the discount rate d being somewhere in the 2-6\% range}$$