Handout 1

CE vs. PO

Competitive Equilibrium

Definition*: A competitive equilibrium* for the economy is an allocation of goods among consumers and production programs for all firms and a supporting price vector such that all demands and supplies are equilibrated.

A. For the Firm:
   1. p denotes the price vector (1 x N)
   2. y denotes the firm's production vector (N x 1).
      a. If $y_i < 0$, it is an input.
      b. If $y_i > 0$, it is an output.
   3. $f(y) \leq 0$ is the firm's implicit production function.
   4. Maximize py subject to $f(y) \leq 0$:

$$L(y,\delta) = py - \delta \cdot f(y)$$

$$\partial L / \partial y_i = p_i - \delta \cdot \partial f / \partial y_i = 0 \qquad \text{for all i}$$

$$p_i = \delta \cdot \partial f / \partial y_i \qquad \text{for all i}$$

Also, for good k: $p_k = \delta \cdot \partial f / \partial y_k$

   5. By division,

$$\frac{p_i}{p_k} = \frac{\partial f / \partial y_i}{\partial f / \partial y_k}$$

$f(y_-, y_k) = 0$ [y_- is the vector y with element y_k removed.]

Implicit Function Theorem $\Rightarrow y_k = y_k(y_-)$

$$f(y_-, y_k(y_-)) = 0$$

$$\frac{\partial f}{\partial y_i} + \frac{\partial f}{\partial y_k}\frac{\partial y_k}{\partial y_i} = 0$$

$$\frac{\partial y_k}{\partial y_i} = -\frac{\partial f / \partial y_i}{\partial f / \partial y_k}$$

   6. Therefore,

$$\frac{p_i}{p_k} = -\frac{\partial y_k}{\partial y_i} \qquad \text{for all i, k}$$

   7. Classical conditions obtainable from FOC in 4 above:
      a. If y_k is an output and y_i is an input, then $\frac{p_i}{p_k}$ = MP.

b. If y_k and y_i are outputs, then $\frac{p_i}{p_k}$ = MRT (sometimes MRPT).

c. If both are inputs, then $\frac{p_i}{p_k}$ = MRTS.

8. Major assumptions to obtain above FOC:
   a. Profit maximization
   b. $y_i^* \neq 0$
   c. Second-order conditions
      i. py is concave (obvious)
      ii. f(y) is concave

9. Expand for many firms:
   a. Suppose there are m firms indexed j = 1, 2,…, m.
   b. $f^j\left(y^j\right) \leq 0$ for all j
   c. The following FOC results from profit maximization for all firms and goods:

$$p_i = \delta^j \frac{\partial f^j}{\partial y_i^j} \quad \text{for all i, j} \qquad [1]$$

10. Resulting profits are denoted π^j; combine into mx1 vector = π

B. For the Consumer:

1. Notation:

   H individuals

   x^h: consumption bundles $(N \times 1)$; $x_i^h \geq 0$

   U^h: utility function

   ω^h: initial endowments (N x 1)

   θ^h: shareholdings of firms (1 x m)

   p: price vector (1 x N)

2. Utility Maximization:

   Maximize $U^h\left(x^h\right)$ subject to $px^h \leq p\omega^h + \theta^h\pi$

$$L^h\left(x^h, \gamma^h\right) = U^h\left(x^h\right) - \gamma^h\left(px^h - p\omega^h - \theta^h\pi\right)$$

$$\frac{\partial L^h}{\partial x_i^h} = \frac{\partial U^h}{\partial x_i^h} - \gamma^h p_i = 0 \quad \text{for all h, i}$$

$$\frac{\partial U^h}{\partial x_i^h} = \gamma^h p_i \quad \text{for all h, i} \qquad [2]$$

3. Interpretation:

$$\frac{\frac{\partial U^h}{\partial x_i^h}}{\frac{\partial U^h}{\partial x_k^h}} = \frac{p_i}{p_k}$$

$$MRS = \frac{p_i}{p_k}$$

4. Major Assumptions:
   a. Utility maximization
   b. $x_i^{h*} > 0$
   c. Second-order conditions
      i. U^h is concave
      ii. $px^h - p\omega^h + \theta^h \pi$ is convex

Pareto Optimality

Definition: A *Pareto Optimum* is a state of the economy in which there are no alternative allocations of production or consumption bundles where everyone is at least as well off and at least one agent is strictly better off.

A. This definition is suggestive of an optimization problem.

B. Let $\omega_i = \sum_{h=i}^{H} \omega_i^h$ as a small notational convenience to be used in (D) below.

C. Then, maximize $U^g\left(x^g\right)$ subject to

$$U^h\left(x^h\right) \geq \bar{U}^h \qquad \text{for all } h \neq g,\ 1, 2,..., H$$

$$f^j\left(y^j\right) \leq 0 \qquad \text{for all } j = 1, 2,..., m$$

$$\sum_{h=1}^{H} x_i^h \leq \omega_i + \sum_{j=1}^{m} y_i^j \qquad \text{for all } i = 1, 2,..., N$$

D. $$L(...) = U^g\left(x^g\right) + \sum_{\substack{h=1\\h\neq g}}^{H} \lambda^h\left(U^h\left(x^h\right) - \bar{U}^h\right) - \sum_{j=1}^{m} \alpha^j f^j\left(y^j\right) + \sum_{i=1}^{N} \beta_i\left(\omega_i + \sum_{j=1}^{m} y_i^j - \sum_{h=1}^{H} x_i^h\right)$$

Define $\lambda^g = 1$ and $\bar{U}^g = 0$.

E. $$L(...) = \sum_{h=1}^{H} \lambda^h\left(U^h\left(x^h\right) - \bar{U}^h\right) - \sum_{j=1}^{m} \alpha^j f^j\left(y^j\right) + \sum_{i=1}^{N} \beta_i\left(\omega_i + \sum_{j=1}^{m} y_i^j - \sum_{h=1}^{H} x_i^h\right)$$

The important choice variables are y^j and x^k.

y_i^j : $-\alpha^j \frac{\partial f^j}{\partial y_i^j} + \beta_i = 0$ for all i,j implies

$$\beta_i = \alpha^j \frac{\partial f^j}{\partial y_i^j} \quad \text{for all i,j} \qquad [3]$$

x_i^h : $\lambda^h \frac{\partial U^h}{\partial x_i^h} - \beta_i = 0$ for all i,h implies

$$\beta_i = \lambda^h \frac{\partial U^h}{\partial x_i^h} \quad \text{for all i,h} \qquad [4]$$

F. Recite a common PO condition. It can be easily shown using [3] and/or [4].

Comparing CE and PO

A. Equations [1] and [2] are equivalent to [3] and [4] if:

$\alpha^j = \delta^j$ for all j

$1/\lambda^h = \gamma^h$ for all h

$\beta_i = p_i$ for all i

B.

Social	Private	Concept
α^j	δ^j	marginal value of technological constraints
$1/\lambda^h$	γ^h	marginal utility of income
β_i	p_i	marginal value of a commodity

Appendix: Kuhn-Tucker Methodology

Problem:

Max $V(\underline{x})$ subject to $h(\underline{x}) \le a$

Restate constraint as: $a - h(\underline{x}) \ge 0$.

Construct the following Lagrangian:

$$L(\underline{x},\lambda) = V(\underline{x}) + \lambda \cdot (a - h(\underline{x}))$$

Necessary Conditions

(1) $\frac{\partial V}{\partial x_i} - \lambda \cdot \frac{\partial h}{\partial x_i} \le 0$ for all i

(2) $\left(\frac{\partial V}{\partial x_i} - \lambda \cdot \frac{\partial h}{\partial x_i}\right) \cdot x_i = 0$ for all i

(3) $a - h(\underline{x}) \ge 0$

(4) $(a - h(\underline{x})) \cdot \lambda = 0$

Sufficient Conditions

(1) $V(\underline{x})$ is concave function. This is equivalent to $D^2[V(\underline{x})]$ being negative definite.

(2) $a - h(\underline{x})$ is a convex function. This is equivalent to $D^2[h(\underline{x})]$ being negative definite.

(3) There exists an $\underline{x}$ such that the constraint(s) holds with a strict inequality (this is called constraint qualification).

Definition: $V(\underline{x})$ is a concave function if and only if

$$V(t\underline{x}_1 + (1-t)\underline{x}_2) \ge tV(\underline{x}_1) + (1-t)V(\underline{x}_2) \text{ for } 0 \le t \le 1.$$

$V(\underline{x})$ is convex if $-V(\underline{x})$ is concave.

Simplification:

In economics we normally presume that all constraints are binding and that the solution is interior. Therefore, $\lambda > 0$ and $x_i \ne 0$ for all i. Hence, the K-T necessary conditions reduce to

$$\frac{\partial V}{\partial x_i} - \lambda \cdot \frac{\partial h}{\partial x_i} = 0 \quad \text{for all i}$$

and $a - h(\underline{x}) = 0$.

Minimization:

Reverse the inequalities in the problem specification and the first and third necessary conditions. Also change sufficient conditions (1) and (2). Everything else is okay.