Handout 8

Path Independence for ΔS

A. Many policies will induce changes in more than one price and possibly income, too. The most straight forward method of calculating ΔS for such a multiprice/income policy is to sequentially determine the consumer's ΔS_i for each change and add the results together. This generates the question, "Does the order (path) of sequential evaluation matter?" The term "path independence" means that we obtain the same ΔS regardless of path. This is obviously a desirable property because there is no reason to believe that there is a preferred path, so we would like all selectable paths to yield the same result. It turns out that the conditions necessary to achieve path independence (PI) can be quite restrictive.

B. Necessary conditions for PI are derived below. The applicable situation is that a single consumer will be affected by changing prices and/or income. Some of the affected prices may be for goods supplied by the consumer, such as labor, and from which the consumer derives income. Labor and similar income sources are *endogenous incomes* in that the consumer decides how much of the resource to supply. In addition, the consumer may receive *exogenous income* from passive income like dividends, welfare payments, or lotteries.

C. Notation

- $\underline{x}$: n x 1, goods for which prices may change
- $\underline{c}$: n x 1, endowments for all goods; many c_i may be zero; for leisure you can think of $c_i = 24$ hours; if $x_i - c_i < 0$, this good is a source of endogenous income
- x_{n+1}: numeraire good, constant price = 1, no initial endowment
- $\underline{p}$: 1 x n, price vector
- m: 1 x 1, exogenous income

D. The consumer maximizes $U(\underline{x}, x_{n+1})$ subject to $\sum_{i=1}^{n} p_i(x_i - c_i) + x_{n+1} \le m$.

$$L(\underline{x}, x_n + 1, \lambda) = U(\underline{x}, x_n + 1) - \lambda\left(\sum_{i=1}^{n} p_i(x_i - c_i) + x_{n+1} - m\right)$$

Noteworthy FOC's are $U_i = \lambda p_i \quad i = 1,2,...,n$, and $U_{n+1} = \lambda$.

E. A *line integral* is a generalization of integration where (1) the integration involves more than one dimension of change (e.g., two prices are changing), and (2) the integration is conducted along a prespecified line (path) we shall call L. Using a line integral, by definition,

$$\Delta U = \int_L dU.$$

The total derivative of U is $dU = \sum_{i=1}^{n} U_i dx_i + U_{n+1} dx_{n+1}$.

Substituting,

$$\Delta U = \int_L \left(\sum_{i=1}^{n} U_i dx_i + U_{n+1} dx_{n+1} \right).$$

Substituting the FOC's

$$\Delta U = \int_L \lambda \left(\sum_{i=1}^{n} p_i dx_i + dx_{n+1} \right).$$

The marginal utility of income, λ, is unknown to the analyst. If it were constant along L, we could factor it out and easily deal with the remaining integral. What we really do is give up on measuring ΔU and focus on ΔS instead where

$$\Delta S = \int_L \left(\sum_{i=1}^{n} p_i dx_i + dx_{n+1} \right).$$

[Notice the difference between the last two equations.]

Totally differentiate the budget constraint to obtain

$$\sum p_i dx_i + \sum (x_i - c_i) dp_i + dx_{n+i} = dm.$$

By substitution,

$$\Delta S = \int_L \left(\sum (c_i - x_i) dp_i + dm \right).$$

[$c_i - x_i$ is the net supply of good i offered by the consumer. If $c_i = 0$, x_i is a pure demand.]

This last expression for ΔS typifies our usual approach to surplus estimation over a set of price/income changes. For price changes we integrate over the price range. The same applies to wage changes. Changes in exogenous income are simply measured by the income change.

F. The big issue is "Does the resulting ΔS depend on the choice of L?"

The applicable mathematics theorem is that $\int_L \sum_i g_i(\underline{z}) dz_i$ is independent of L if $\dfrac{\partial g_i}{\partial z_j} = \dfrac{\partial g_j}{\partial z_i}$ for all i, j included in the summation. Applying this theorem to the preceding expression for ΔS, we obtain the requirements

$$\frac{\partial x_i}{\partial p_j} = \frac{\partial x_j}{\partial p_i} \quad \text{for all goods for which price is changing.} \qquad (*)$$

and $\dfrac{\partial x_i}{\partial m} = \dfrac{\partial m}{\partial p_i} = 0$ for all goods for which price is changing if m is also changing. (**)

Note that these requirements only apply if the policy change impacts certain prices and exogenous income. If, for example, the policy affects prices and wages but not exogenous income, then (**) is unnecessary for PI.

G. Further refinement of (*) into a more readily applied form requires the Slutsky equation, but the usual Slutsky equation does not hold for the net demand/supply case (with initial endowment), so we now generate the more general Slutsky equation where income is composed of exogenous income (m) and endogenous income:

Differentiate $x_i^*(\underline{p}, M)$ recognizing that $M = m + \underline{p} \bullet \underline{c}$

$$\frac{\partial x_i^*}{\partial p_j} = \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial M}\frac{\partial M}{\partial p_j} = \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial M} c_j.$$

Now substitute the usual Slutsky eq. for the first term on the right and collect terms:

$$\frac{\partial x_i^*}{\partial p_j} = \frac{\partial h_i}{\partial p_j} - x_j \frac{\partial x_i}{\partial M} + \frac{\partial x_i}{\partial M} c_j$$

$$\frac{\partial x_i^*}{\partial p_j} = \frac{\partial h_i}{\partial p_j} + (c_j - x_j)\frac{\partial x_i}{\partial M}. \qquad \text{[The general Slutsky equation]}$$

H. Suppose condition (∗) holds and substitute in general Slutsky equations:

$$\frac{\partial h_i}{\partial p_j} + (c_j - x_j)\frac{\partial x_i}{\partial m} = \frac{\partial h_j}{\partial p_i} + (c_i - x_i)\frac{\partial x_j}{\partial m}$$

The first two terms of each side are always equal, so

$$(c_j - x_j)\frac{\partial x_i}{\partial m} = (c_i - x_i)\frac{\partial x_j}{\partial m}$$

Modifying so that income elasticities are explicit:

$$(c_j - x_j)\frac{\partial x_i / x_i}{\partial m/m} \bullet \frac{x_i}{m} = (c_i - x_i)\frac{\partial x_j / x_j}{\partial m/m} \bullet \frac{x_j}{m}$$

$$\frac{c_j - x_j}{x_j} \bullet \frac{\partial x_i / x_i}{\partial m/m} = \frac{c_i - x_i}{x_i} \bullet \frac{\partial x_j / x_j}{\partial m/m}$$

$$\frac{c_j - x_j}{x_j} \bullet \eta_i = \frac{c_i - x_i}{x_i} \bullet \eta_j \qquad (***)$$

which is a relation between income elasticities that can be checked.

If there are no initial endowments for goods i and j, then this reduces to

$$\eta_i = \eta_j. \qquad (****)$$

<u>Two Conclusions (for when **some** prices change)</u>

1. The above steps are reversible, so we get PI condition (∗) if and only if (***) holds for all goods experiencing policy-induced price/wage changes. This assumes that exogenous income is not being altered by the policy.

2. If exogenous income is also changing as a consequence of policy, then (∗∗) obtains if and only if

$$\eta_i = 0 = \eta_j.$$

I. Clearly the conditions necessary for PI are rather rigid.

J. The entire development above presumed a numeraire for which price is unaffected by the policy. This seems very satisfactory since the consumer would presumably only care about relative price changes if all prices are to be altered by policy.

The above analysis is, however, easily extended to price changes for <u>all</u> goods. (*) and (**) again apply as do the resulting requirements on exogenous income elasticities. Two additional results can be generated.

<u>More Conclusions (for when **all** prices change)</u>

3. Suppose that m is unaltered by the policy, so that only (*) is needed for PI. Suppose that (*) is satisfied because there are no initial endowments and all income elasticities are equal. Let's differentiate the budget constraint with respect to m and substitute $\eta_i = \eta$ for all i.

$$\sum_{i=1}^{n+1} p_i x_i = m$$

$$\sum_{i=1}^{n+1} p_i \frac{\partial x_i}{\partial m} = 1$$

$$\sum_{i=1}^{n+1} p_i \frac{x_i}{m} \eta_i = 1$$

$$\eta \sum_{i=1}^{n+1} p_i x_i = m$$

$$\eta \bullet m = m$$

$$\eta = 1$$

Therefore, for PI in the case of all prices changing, no initial endowments, and constant exogenous income, $\eta = 1$ is necessary.

4. If m is policy modified as well as <u>all</u> prices, then PI is impossible because $\eta_i = 0$ for all i would violate the budget constraint. [We cannt have more/less income without at least one demanded quantity changing.]

K. Summary

To check if your analytical situation exposes you to path dependency, which of these four possible scenarios applies?

1. Not all prices are changing, and exogenous income is constant.
2. Not all prices are changing, and exogenous income is changing too.
3. All prices are changing, and exogenous income is constant.
4. All prices are changing, and exogenous income is changing too.

These four scenarios are each separately covered by the four enumerated and bolded items explored above.

L. Homotheticity

Homothetic utility functions have the property that income elasticities for all goods are equal and are given by (∗∗∗∗). Therefore, homothetic utility functions satisfy the conditions necessary for PI as long as exogenous income is not changing and the consumer is not endowed with those goods for which price is changing.

References

Just, Richard E., Darrell L. Hueth, and Andrew Schmitz. *The Welfare Economics of Public Policy*. Northampton, MA: Edward Elgar Publishing, Inc., 2004.