Handout 9

Methods of Obtaining "Exact" Welfare Measures

There are various approaches for estimating consumer welfare using either Marshallian or Hicksian measures. Figure 1 summarizes the known methods. While Hicksian measures (CV, EV, CS, ES) have long been acknowledged as appropriate for welfare analysis, Marshallian measures have dominated applied work. The reason lies in the difficulty of obtaining Hicksian measures in the real world; utility functions are difficult to observe. There are, however, circumstances in which *exact* or Hicksian measures can be determined. These circumstances are somewhat confining, but there are policy issues which meet the needed conditions. This handout explains the method of obtaining exact measures using ordinary demand functions (2B in Figure 1).

The essential ingredients for understanding how to obtain exact welfare measures are, in addition to the material summarized in the earlier handout on consumer welfare measurement, as follows: (a) Roy's Identity, (b) an application of the Implicit Function Theorem, and (c) some knowledge of differential equations.

a. Roy's Identity: $x_n(p,M) = -\dfrac{\dfrac{\partial v(p,M)}{\partial p_n}}{\dfrac{\partial v(p,M)}{\partial M}}$

where x is ordinary demand and v is the indirect utility function. **p** is 1 x N.

b. Consider any indifference surface (or hypersurface) in (**p**, M) space (N + 1 dimensions), $v\left(\mathbf{p},M\left(\mathbf{p}\right)\right) = \bar{U}$. If we assume all prices but one, say p_n, are fixed, we can vary M and p_n so as to stay on the indifference surface. Utility is constant along this two dimensional curve. Taking the total derivative,

$$\frac{\partial v}{\partial p_n} + \frac{\partial v}{\partial M}\frac{dM}{dp_n} = 0,$$

or

$$\frac{dM}{dp_n} = \frac{-\dfrac{\partial v\left(\mathbf{p},M\right)}{\partial p_n}}{\dfrac{\partial v\left(\mathbf{p},M\right)}{\partial M}}$$

Therefore,

$$\frac{dM}{dp_n} = x_n\left(\mathbf{p},M\right). \qquad (1)$$

This is a differential equation. If it can be solved, we will have recovered enough information about the expenditure function to compute either CV or EV. The right hand side of (1) is merely Marshallian demand. The procedure is presumably to collect market data, fit the data to some functional form, solve the above differential equation after substituting the estimated demand function, and rearrange the result to allow computation of

CV or EV.

c. The form of the solution to the differential equation depends almost entirely upon the form of the estimated demand function. In fact, not all differential equations have analytically determinable solutions, so functional form for the demand relation must be chosen purposefully. For present purposes one noncritical assumption is made about the demand function: p_n is the only price variable.[1] The differential equation is now

$$\frac{dM}{dp_n} = x_n\left(p_n, M\right). \tag{2}$$

The solution to this equation will define a family of curves which differ by an arbitrary constant.

For example, suppose ordinary demand is precisely known as

$$\frac{dM}{dp_n} = \frac{M}{p_n} \Rightarrow \frac{dM}{M} = \frac{dp_n}{p_n}.$$

[There could be a β on the right hand side, but it would not change the procedure.] Integrating both sides yields

$$\ln(M) = \ln(p_n) + \ln(k)$$

where k is an arbitrary constant. Rearranging yields

$$M = p_n k.$$

We can solve for the constant if we have additional information. Often we do. For example, income and postpolicy price might be known. Plugging them in above would provide "subsequent" k. The argument below indicates that this subsequent k is interpretable as subsequent utility. Plugging into the above equation provides the expenditure function with subsequent utility already entered. EV can be immediately computed.

If we include the arbitrary constant as an argument we have the following solution to (2).

$$M = f\left(p_n, k\right) \tag{3}$$

Two further points will complete the problem. First, choice of the arbitrary constant selects the indifference surface to which all of this pertains. At the beginning of b we selected utility level $\bar{U}$. k is a function of $\bar{U}$ only, and $\bar{U}$ can replace k in the last equation. If $\bar{U}$ is the initial utility level, the above relation can be used to compute CV. If $\bar{U}$ is the subsequent utility level, we can compute EV. Second, equation (3) is the expenditure function with all prices but one held fixed. Because the differential equation (1) defined a curve on an indifference surface, M is the minimum expenditure necessary to achieve utility level $\bar{U}$ when the price of good n is p_n. Therefore,

$$M = e\left(p_n, \bar{U}\right)$$

$$CV = e\left(p_n^0, U_0\right) - e\left(p_n^1, U_0\right)$$

[1] This assumption is for notational convenience only. As long as the purpose of all this is to get an exact welfare measure for a change in p_n only, other prices may appear in the estimated demand function.

$$EV = e\left(p_n^0, U_1\right) - e\left(p_n^1, U_1\right)$$

The significance of these results is that Hicksian measures can be obtained from market data, at least in the case where only one price is changing.

An Example with a Linear Specification for Marshallian Demand

To obtain an exact welfare measure for a change in one price, it is assumed that all other prices except the one in question are fixed and that these goods with fixed prices are treated as a Hicksian composite commodity with a single price. This procedure reduces an n-good case to a simple two-good situation where only one price is changing. Consider the following linear demand functional form for equation (2):

$$x = \alpha + \beta M + \gamma P \tag{4}$$

where α, β, and γ are parameters, P is the price of the good in question and M is the income, both of which are normalized by the price of the Hicksian composite commodity. By solving the first order differential equation of (2) with this linear demand functional specification [something you are not required to know how to do], a corresponding expenditure function is obtained as

$$e\left(P,u\right) = M = -\frac{\gamma\left(\beta P + 1\right)}{\beta^2} - \frac{\alpha}{\beta} + k \bullet \exp\left(\beta P\right) \tag{5}$$

where $\exp\left(x\right) = e^x$. Here, e is not the expenditure function; it is the constant given by ln(1).

Once the expenditure function is derived, calculating CV or EV is quite simple. For the actual values of CV and EV, the expenditure function of equation (5) needs to be solved for utility-giving indirect utility function,

$$v\left(P,M\right) = u = \left(M + \frac{\gamma\left(\beta P + 1\right)}{\beta^2} + \frac{\alpha}{\beta}\right)\exp\left(-\beta P\right). \tag{6}$$

Then,

$$CV = e\left(P^\circ, u^\circ\right) - e\left(P^1, u^\circ\right) \tag{7}$$

$$= -\frac{\gamma P^\circ}{\beta} - \frac{\gamma}{\beta^2} - \frac{\alpha}{\beta} + \frac{\gamma P^1}{\beta} + \frac{\gamma}{\beta^2} + \frac{\alpha}{\beta} + u^\circ\left(\exp(\beta P^\circ) - \exp(\beta P^1)\right)$$

$$= -\frac{\gamma}{\beta}\left(P^\circ - P^1\right) + u^\circ\left(\exp(\beta P^\circ) - \exp(\beta P^1)\right) \tag{8}$$

The u° term of equation (8) can now be determined via equation (6) and substituted into (8) to fully determine CV.

The derivation of EV is parallel and only involves a modification in the very last step.

A study by Chang and Griffin investigated alternative functional forms for Marshallian demand to see whether they could be used in this type of work. Of the fifteen functional specifications examined, nine forms are found to have solutions to the first order differential equation, and these are listed in Table 1. Estimation of variation measures using Marshallian approximation is unnecessary if one of these nine functional forms is used.

Additional Notes

A. Finding exact welfare measures for multiple price changes is certainly a more useful endeavor. There is some very technical literature on this topic. In the case of multiple price changes, equation (2) becomes a system of partial differential equations which must be solved simultaneously. For example, if price is changing for goods 1 and 2, equation 2 becomes

$$\frac{\partial M}{\partial p_1} = x_1\left(p_1, p_2, M\right)$$
$$\frac{\partial M}{\partial p_2} = x_2\left(p_1, p_2, M\right).$$

This type of problem can be very difficult to solve, either analytically or numerically. The first query is, as always, whether a solution even exists. Employing the Frobenius Theorem (Hurwicz, Varian), we have existence if $\frac{\partial x_i}{\partial p_j} = \frac{\partial x_j}{\partial p_i}$ for all i, j for which price is changing.

These are the so-called integrability conditions; they imply that the system can be "integrated" to generate utility, indirect utility, and expenditure functions. It is notable that these conditions are the same as the Slutsky conditions which result from "well behaved" utility functions. Because the integrability conditions are necessary for existence of the solution to the partial differential equation system, they must be imposed when econometrically estimating systems of demand functions. Even then, a solution will be difficult to obtain unless functional form is purposefully selected.

B. Just, Hueth, and Schmitz offer an interesting suggestion (pp. 177-82) which can eliminate the burden of solving the differential equation or system of differential equations. [I'm not sufficiently knowledgeable about the literature to know where this idea originated.]

- Choose a functional form for the utility function, and use this to derive forms for Marshallian demand and the expenditure function.
- Estimate the demand function using market data.
- Estimated coefficients can be substituted into the expenditure function so that it is now known and exact measures can be computed.

Intuitively, the applicability of this method depends on the choice of functional form for the utility function. This method has the advantage of making *exact* welfare measurement for multiple price changes much more clear cut.

References

Chang, Chan and Ronald C. Griffin. "Demand Functions for Exact Welfare Measurement." Selected paper presented at *American Agricultural Economics Association*. Baltimore, Maryland, August 9-12, 1992.

Hausman, Jerry A. "Exact Consumer's Surplus and Deadweight Loss." *American Economic Review* 71 (September 1981): 662-76.

Hurwicz, Leonid. "On the Problem of Integrability of Demand Functions." In *Preferences, Utility, and Demand*, pp. 174-214, edited by John S. Chipman et al.. New York: Harcourt, Brace Jovanovich, Inc., 1971.

Just, Richard E., Darrell L. Hueth, and Andrew Schmitz. *The Welfare Economics of Public Policy: A Practical Approach to Policy and Project Evaluation*. Northampton, MA: Edward Elgar, 2004.

Samuelson, Paul A. "The Problem of Integrability in Utility Theory." *Economica* 17 (November 1950): 355-85.

Shonkwiler, J.S. "Consumer's Surplus Revisited." *American Journal of Agricultural Economics* 73 (1991): 410-414.

Varian, Hal R. *Microeconomic Analysis*. New York: W. W. Norton and Company, Inc., 1978.

Willig, Robert. " Consumer's Surplus Without Apology." *American Economic Review* 66 (1976): 589-97.

Table 1. Functional Forms Available for Exact Welfare Measurement

Name	Functional Form
1. Linear	$x = \alpha + \beta M + \gamma P$
2. Log-Linear	$x = e^{\alpha} P^{\beta} M^{\gamma}$
3. Semi-Log	$ln(x) = \alpha + \beta P + \gamma M$
4. Quadratic	$x=\alpha+\beta M+\gamma P+\delta M^2 + \phi P^2 + \eta MP$
5. Double-Log	$x = \alpha M^{\beta} P^{\gamma}$
6. Mitscherlich	$x = \alpha + \{1 - \exp(\beta M)\}\{1 - \exp(\gamma P)\}$
7. Spillman	$x=\alpha(1-\beta^{M})(1-\gamma^{P})$
8. Resistance	$x^{-1} = \alpha + \beta(\gamma + M)^{-1} + \delta(\phi + P)^{-1}$
9. Inverse	$x = \alpha + \beta/M + \gamma/P$

All forms correspond with the case of two goods (one good in question and one Hicksian composite commodity) and one changing price, while both price (P) and income (M) are deflated by the price of the Hicksian composite commodity.

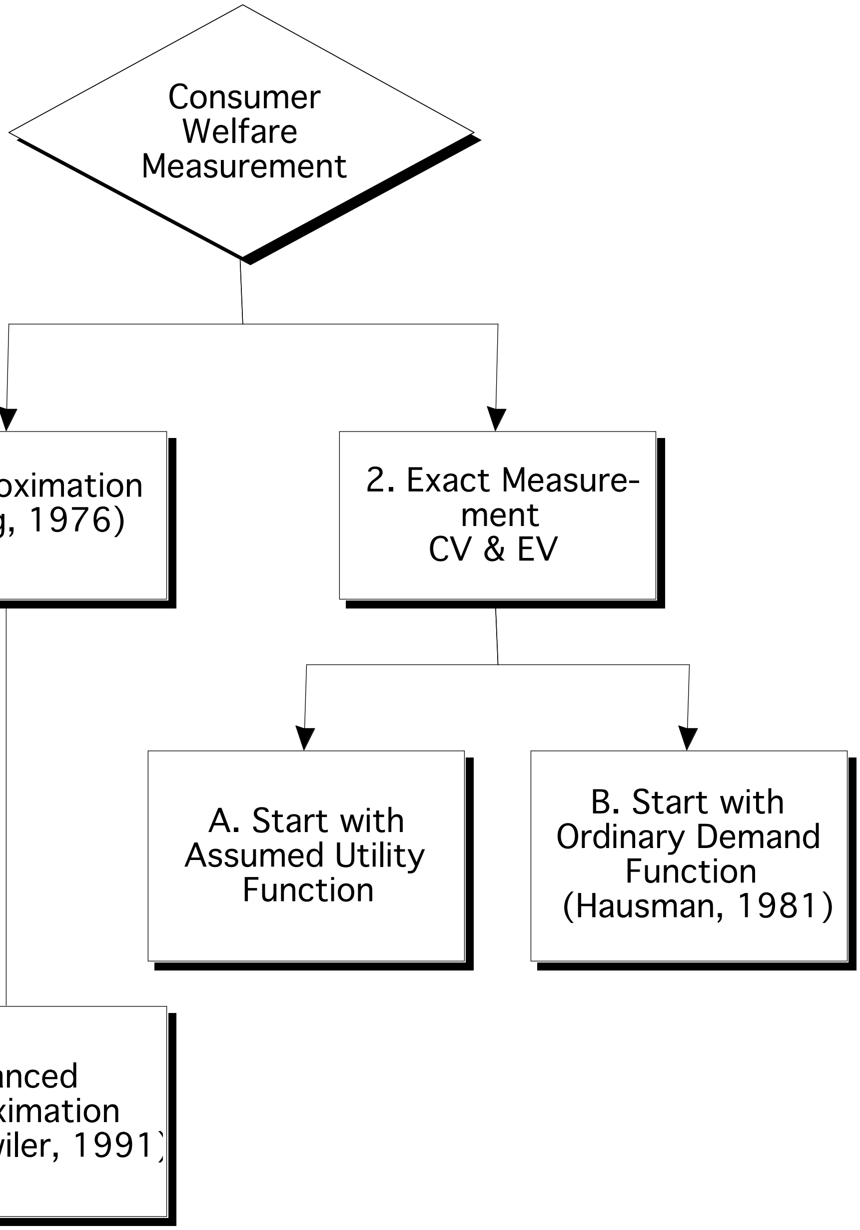

Figure 1. Methods of Consumer Welfare Measurement